J. Math. Anal. Appl. 376 (2011) 317–328

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On a nonlinear generalized max-type difference equation

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ARTICLE INFO

Article history:
Received 4 December 2009
Available online 19 November 2010
Submitted by R. Curto

Keywords:
Positive solution
Difference equation
Boundedness
Max-type

ABSTRACT

The boundedness character of positive solutions of the following max-type difference equation

$$x_n = \max\left\{A, \frac{x_{n-1}^p}{x_{n-k}^r}\right\}, \quad n \in \mathbb{N}_0,$$

where $k \in \mathbb{N} \setminus \{1\}$, the parameters A and r are positive and p is a nonnegative real number is studied in this paper. Our main results considerably improve results appearing in the literature.

1. Introduction

Recently there has been a great interest in studying nonlinear difference equations for developing some techniques which can be used in investigating the equations arising in models describing real life situations in biology, control theory, economics, etc. (see, e.g., [2,9,12,16–18] and the references therein).

The following max-type difference equation

$$x_n = \max\left\{A, \frac{x_{n-m}^p}{x_{n-k}^r}\right\}, \quad n \in \mathbb{N}_0, \tag{1}$$

where the parameters A and r are positive and p is a nonnegative real number and $k, m \in \mathbb{N}$, $k \neq m$ is a quite general difference equation whose behavior is complicated. It is a basic nonlinear difference equation containing a nonrational term, which generalizes a max-type difference equation with a rational term. Beside this, Eq. (1) is a good prototype for investigating max-type difference equations.

A special case of the following difference equation

$$y_n = \max\left\{\frac{A}{y_{n-1} \cdots y_{n-k+1}}, \frac{1}{y_{n-k-1} \cdots y_{n-2k+1}}\right\}, \quad n \in \mathbb{N}_0$$

(with $k = 2$) arises naturally in certain models in control theory (see, [17]). By the change $x_n = y_n y_{n-1} \cdots y_{n-k+1}$ the equation is transformed into the equation

$$x_n = \max\left\{A, \frac{x_{n-1}}{x_{n-k}}\right\}, \quad n \in \mathbb{N}_0,$$

which is a special case of Eq. (1). Some results on max-type difference equations can be found, e.g., in [1,4,6,8,9,13–15,22, 24–30] (see also numerous references therein).

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.11.041

Here we study the boundedness character of positive solutions of the following difference equation

$$x_n = \max\left\{A, \frac{x_{n-1}^p}{x_{n-k}^r}\right\}, \quad n \in \mathbb{N}_0, \tag{2}$$

where $k \in \mathbb{N} \setminus \{1\}$, the parameters A and r are positive and p is a nonnegative real number. This paper can be considered as a continuation of our previous investigations, see, for example, [24,25].

Studying the boundedness character of a difference equation is important since it is a basic precondition for the stability or periodicity of all solutions of the equation.

In [14] was proved that positive solutions to the equation

$$y_n = \max\left\{A, \frac{y_{n-1}}{y_{n-2}}\right\}, \quad n \in \mathbb{N}_0, \tag{3}$$

are eventually periodic, so bounded. An elegant proof of this fact was essentially given in [19]. We would like to point out that some of our theorems are motivated by this proof. We want to present the proof of the result for the benefit of the reader as well as, as a basic source from which the proofs of some of our theorems stem from. Indeed, from (3) it follows that $y_n \geqslant A$ for all $n \in \mathbb{N}_0$. On the other hand, we have that

$$\begin{aligned} y_n &= \max\left\{A, \frac{y_{n-1}}{y_{n-2}}\right\} \\ &= \max\left\{A, \frac{A}{y_{n-2}}, \frac{y_{n-2}}{y_{n-2}y_{n-3}}\right\} \\ &= \max\left\{A, \frac{A}{y_{n-2}}, \frac{1}{y_{n-3}}\right\} \\ &\leqslant \max\left\{A, \frac{A}{A}, \frac{1}{A}\right\} = \max\left\{A, \frac{1}{A}\right\}, \end{aligned}$$

$n \geqslant 3$, finishing the proof of the result.

Note that the case $p = 0$ is trivial, since from the obvious inequality $x_n \geqslant A$, $n \in \mathbb{N}_0$, for every solution of Eq. (2) we have that

$$x_n \leqslant \max\left\{A, \frac{1}{A^r}\right\}, \quad n = k, k+1, \ldots,$$

from which the boundedness immediately follows.

Nowadays there is a great interest in studying difference equations whose all positive solutions are away from zero, see, for example, [3,5,7,9–11,19–23] and the references therein.

2. Boundedness of Eq. (2)

The boundedness character of positive solutions of Eq. (2) is studied in this section. Several cases will be considered separately.

2.1. Case $p^k \geqslant \max\{r\frac{k^k}{(k-1)^{k-1}}, \frac{k^k}{(k-1)^k}\}$ or $r < p - 1 \leqslant \frac{1}{k-1}$

Here we investigate Eq. (2) for the case in the title. By making use of a connection of Eq. (2) with a linear difference inequality with constant coefficients in the first result of this paper we prove that Eq. (2) has positive unbounded solutions in the case. Namely, we prove the following result:

Theorem 1. *Assume that*

$$p^k \geqslant r\frac{k^k}{(k-1)^{k-1}} \quad \text{and} \quad p \geqslant \frac{k}{k-1} \tag{4}$$

(*where at least one of these two inequalities is strict*) *or $r < p - 1 < \frac{1}{k-1}$. Then Eq.* (2) *has positive unbounded solutions.*

Proof. First, note that for every solution to Eq. (2) the following inequality holds

$$x_n \geqslant \frac{x_{n-1}^p}{x_{n-k}^r}, \quad n \in \mathbb{N}_0, \tag{5}$$

as well as that it is bounded below by A for $n \in \mathbb{N}_0$.

Set

$$y_n = \ln x_n.$$

Taking the logarithm of (5), it follows that

$$y_n - py_{n-1} + ry_{n-k} \geqslant 0, \quad n \in \mathbb{N}_0. \tag{6}$$

Let

$$P(\lambda) = \lambda^k - p\lambda^{k-1} + r.$$

Then, we have

$$P'(\lambda) = k\lambda^{k-1} - (k-1)p\lambda^{k-2},$$

from which it follows that the polynomial $P(\lambda)$ has a local minimum at the point

$$\lambda_k = \frac{(k-1)p}{k},$$

and

$$P(\lambda_k) = \frac{(k-1)^{k-1}}{k^k}\left(r\frac{k^k}{(k-1)^{k-1}} - p^k\right) \leqslant 0. \tag{7}$$

If $p > k/(k-1)$, then $(k-1)p/k > 1$. From this, (7) and $\lim_{\lambda\to\infty} P(\lambda) = +\infty$, it follows that there is a $\lambda_1 > 1$ such that $P(\lambda_1) = 0$. If $p = k/(k-1)$, according to the assumptions, inequality (7) is strict, $(k-1)p/k = 1$, and we also have that there is a $\lambda_1 > 1$ such that $P(\lambda_1) = 0$.

Now assume that $r < p - 1 < 1/(k-1)$. Then $P(1) = 1 - p + r < 0$ and we again have that there is a $\lambda_1 > 1$ such that $P(\lambda_1) = 0$.

Now note that inequality (6) can be written in the following form

$$P_1\big(P_2(y_n)\big) \geqslant 0, \tag{8}$$

where

$$P_1(u_n) = u_n - \lambda_1 u_{n-1},$$

λ_1 is a real root greater than one, whose existence has been proved above, and $u_n = P_2(y_n)$, where P_2 is the linear operator obtained by the polynomial $P_2 = P/P_1$, that is, if

$$P(\lambda) = P_1(\lambda)P_2(\lambda) = (\lambda - \lambda_1)\big(\lambda^{k-1} + c_{k-2}\lambda^{k-2} + \cdots + c_1\lambda + c_0\big),$$

then

$$u_n = P_2(y_n) = y_n + c_{k-2}y_{n-1} + \cdots + c_1 y_{n-k+2} + c_0 y_{n-k+1}.$$

To see this, notice that the characteristic polynomials associated with the following linear difference equations of order k

$$P_1\big(P_2(y_n)\big) = 0 \quad \text{and} \quad y_n - py_{n-1} + ry_{n-k} = 0$$

are the same.

We can choose initial conditions such that

$$u_{-1} = P_2(y_{-1}) > 0, \tag{9}$$

that is

$$P_2(y_{-1}) = y_{-1} + c_{k-2}y_{-2} + \cdots + c_1 y_{-k+1} + c_0 y_{-k} > 0.$$

Indeed it is clear that we can choose y_{-1} so big that condition in (9) holds.

Inequality (8) can be written as follows

$$u_n - \lambda_1 u_{n-1} \geqslant 0. \tag{10}$$

From (9) and by iterating inequality (10), it follows that

$$u_n \geqslant \lambda_1^{n+1} u_{-1}. \tag{11}$$

Since $\lambda_1 > 1$ and $u_{-1} > 0$, by letting $n \to \infty$ in (11), we obtain $u_n \to \infty$, i.e.

$$\lim_{n\to\infty}(y_n + c_{k-2}y_{n-1} + \cdots + c_1 y_{n-k+2} + c_0 y_{n-k+1}) = +\infty.$$

From this and since the sequence y_n is bounded below with $\ln A$, it follows that there is a subsequence y_{n_k} such that

$$\lim_{k\to\infty} y_{n_k} = +\infty.$$

Otherwise, y_n would be bounded above and consequently bounded which would imply the boundedness of u_n, a contradiction. Therefore

$$\lim_{k\to\infty} x_{n_k} = +\infty,$$

so that x_n is unbounded, as desired. □

Remark 1. From the proof of Theorem 1 we see that the result holds also for all difference equations which satisfy inequality (5) and whose all terms are bounded below.

2.2. *Case $p = r + 1$, $r < \frac{1}{k-1}$*

The next theorem concerns the case $p = r + 1$, $r \in (0, \frac{1}{k-1})$.

Theorem 2. *Assume that $p = r + 1$ and $r \in (0, \frac{1}{k-1})$. Then every positive solution of Eq.* (2) *is bounded.*

Proof. First note that by the change of variable $x_n = Ay_n$, Eq. (2) in the case $p = r + 1$, is transformed into the equation

$$y_n = \max\left\{1, \frac{y_{n-1}^{r+1}}{y_{n-k}^r}\right\}, \quad n \in \mathbb{N}_0.$$

Hence, without loss of generality, we may assume that $A = 1$.

Now we use a new method recently introduced in [25], called *Oachkatzlschwoif* or *Squirrel-tail* method. We have

$$\begin{aligned}
x_n &= \max\left\{1, \frac{x_{n-1}^{r+1}}{x_{n-k}^r}\right\} = \max\left\{1, \frac{x_{n-1}^{a_0^{(1)}}}{x_{n-2}^{a_0^{(2)}} \cdots x_{n-(k-1)}^{a_0^{(k-1)}} x_{n-k}^{a_0^{(k)}}}\right\} \\
&= \max\left\{1, \frac{1}{x_{n-k}^r}, \frac{x_{n-2}^{(1+r)a_0^{(1)} - a_0^{(2)}}}{x_{n-3}^{a_0^{(3)}} \cdots x_{n-(k-1)}^{a_0^{(k-1)}} x_{n-k}^{a_0^{(k)}} x_{n-k-1}^{ra_0^{(1)}}}\right\} \\
&= \max\left\{1, \frac{1}{x_{n-k}^r}, \frac{x_{n-2}^{a_1^{(1)}}}{x_{n-3}^{a_1^{(2)}} \cdots x_{n-(k-1)}^{a_1^{(k-2)}} x_{n-k}^{a_1^{(k-1)}} x_{n-k-1}^{a_1^{(k)}}}\right\} \\
&= \max\left\{1, \frac{1}{x_{n-k}^r}, \frac{1}{x_{n-k}^{a_1^{(k-1)}} x_{n-k-1}^{a_1^{(k)}}}, \frac{x_{n-3}^{(1+r)a_1^{(1)} - a_1^{(2)}}}{x_{n-4}^{a_1^{(3)}} \cdots x_{n-(k-1)}^{a_1^{(k-2)}} x_{n-k}^{a_1^{(k-1)}} x_{n-k-1}^{a_1^{(k)}} x_{n-k-2}^{ra_1^{(1)}}}\right\} \\
&= \max\left\{1, \frac{1}{x_{n-k}^r}, \frac{1}{x_{n-k}^{a_1^{(k-1)}} x_{n-k-1}^{a_1^{(k)}}}, \frac{x_{n-3}^{a_2^{(1)}}}{x_{n-4}^{a_2^{(2)}} \cdots x_{n-k}^{a_2^{(k-2)}} x_{n-k-1}^{a_2^{(k-1)}} x_{n-k-2}^{a_2^{(k)}}}\right\}
\end{aligned}$$

where

$$a_0^{(1)} = 1 + r, \qquad a_0^{(2)} = a_0^{(3)} = \cdots = a_0^{(k-1)} = 0 \quad \text{and} \quad a_0^{(k)} = r.$$

Repeating the procedure for each $n \geqslant l$, we obtain

$$x_n = \max\left\{1, \frac{1}{x_{n-k}^r}, \frac{1}{x_{n-k}^{a_1^{(k-1)}} x_{n-k-1}^{a_1^{(k)}}}, \dots, \frac{x_{n-l}^{a_{l-1}^{(1)}}}{x_{n-l-1}^{a_{l-1}^{(2)}} \cdots x_{n-k-l+1}^{a_{l-1}^{(k)}}}\right\}$$

$$= \max\left\{1, \frac{1}{x_{n-k}^r}, \frac{1}{x_{n-k}^{a_1^{(k-1)}} x_{n-k-1}^{a_1^{(k)}}}, \dots, \frac{1}{x_{n-l-1}^{a_{l-1}^{(2)}} \cdots x_{n-k-l+1}^{a_{l-1}^{(k)}}}, \frac{x_{n-l-1}^{(1+r)a_{l-1}^{(1)} - a_{l-1}^{(2)}}}{x_{n-l-2}^{a_{l-1}^{(3)}} \cdots x_{n-k-l+1}^{a_{l-1}^{(k)}} x_{n-k-l}^{ra_{l-1}^{(1)}}}\right\}. \tag{12}$$

Hence the sequences $a_l^{(i)}$, $i \in \{1, \dots, k\}$ satisfy the system

$$a_l^{(1)} = (1+r)a_{l-1}^{(1)} - a_{l-1}^{(2)}, \qquad a_l^{(i)} = a_{l-1}^{(i+1)}, \quad i \in \{2, \dots, k-1\}, \qquad a_l^{(k)} = ra_{l-1}^{(1)}, \tag{13}$$

and consequently

$$a_l^{(1)} - (1+r)a_{l-1}^{(1)} + ra_{l-k}^{(1)} = 0. \tag{14}$$

We show that the sequences $a_l^{(i)}$, $i \in \{1, \dots, k\}$ are nondecreasing. Moreover, we prove that $a_l^{(1)}$ is increasing. For $l = 1$ this follows from

$$\begin{aligned}
&a_1^{(1)} - a_0^{(1)} = ra_0^{(1)} > 0, \\
&a_1^{(i)} - a_0^{(i)} = a_1^{(i)} = a_0^{(i+1)} = 0, \quad i \in \{2, \dots, k-2\}, \\
&a_1^{(k-1)} - a_0^{(k-1)} = a_1^{(k-1)} = a_0^{(k)} = r > 0, \\
&a_1^{(k)} - a_0^{(k)} = ra_0^{(1)} - r = r^2 > 0.
\end{aligned} \tag{15}$$

Assume that the claim holds for all the indices less than or equal to $l-1$.

By the induction hypothesis, we have

$$\begin{aligned}
&a_l^{(1)} - a_{l-1}^{(1)} = ra_{l-1}^{(1)} - a_{l-1}^{(2)} = r\bigl(a_{l-1}^{(1)} - a_{l-k}^{(1)}\bigr) > 0, \\
&a_l^{(i)} - a_{l-1}^{(i)} = a_{l-1}^{(i+1)} - a_{l-2}^{(i+1)} \geqslant 0, \quad i \in \{2, \dots, k-1\}
\end{aligned}$$

and

$$a_l^{(k)} - a_{l-1}^{(k)} = r\bigl(a_{l-1}^{(1)} - a_{l-2}^{(1)}\bigr) > 0,$$

from which the claim follows.

Now we prove that the sequences $a_l^{(i)}$, $i \in \{1, \dots, k\}$ converge. The characteristic polynomial associated with Eq. (14) is

$$P(\lambda) = \lambda^k - (1+r)\lambda^{k-1} + r = (\lambda - 1)\bigl(\lambda^{k-1} - r\bigl(\lambda^{k-2} + \dots + \lambda + 1\bigr)\bigr).$$

Let

$$f(z) = z^{k-1} \quad \text{and} \quad g(z) = r\bigl(z^{k-2} + \dots + z + 1\bigr).$$

Note that by the condition $r < 1/(k-1)$, we have that on the unit circle $|z| = 1$

$$\bigl|g(z)\bigr| \leqslant r\bigl(|z|^{k-2} + \dots + |z| + 1\bigr) = r(k-1) < 1 = |z|^{k-1} = \bigl|f(z)\bigr|.$$

By Rouche's theorem it follows that the polynomials $f(z)$ and $f(z) - g(z)$ have the same number of zeroes in the unit disk $|z| < 1$. Since $f(z)$ has $k-1$ zeroes in the unit disk, it follows that the polynomial

$$f(z) - g(z) = z^{k-1} - r\bigl(z^{k-2} + \dots + z + 1\bigr),$$

has also $k-1$ zeroes in the unit disk.

Let $\lambda_1, \dots, \lambda_s$ be different zeroes of the polynomial $f - g$, with the multiplicities t_j, $j = 1, \dots, s$. Then

$$a_n^{(1)} = c_1 + \sum_{j=1}^{s} P_j(n)\lambda_j^n,$$

for some $c_1 \in \mathbb{R}$ and some polynomials P_j, $j = 1, \dots, s$. Clearly $a_n^{(1)}$ converges, which along with (13) implies that the sequences $a_l^{(i)}$, $i \in \{2, \dots, k\}$ converge too.

From (12) we have

$$x_n = \max\left\{1, \frac{1}{x_{n-k}^r}, \frac{1}{x_{n-k}^{a_1^{(k-1)}} x_{n-k-1}^{a_1^{(k)}}}, \ldots, \frac{1}{x_{-1}^{a_{n-1}^{(2)}} \cdots x_{-k+1}^{a_{n-1}^{(k)}}}, \frac{x_{-1}^{a_n^{(1)}}}{x_{-2}^{a_n^{(2)}} \cdots x_{-k+1}^{a_n^{(k-1)}} x_{-k}^{a_n^{(k)}}}\right\}. \tag{16}$$

From (16) we have

$$x_n \leqslant \max\left\{1, \frac{1}{x_{k-3}^{a_{n-k+1}^{(2)}} \cdots x_{-1}^{a_{n-k+1}^{(k)}}}, \ldots, \frac{x_{-1}^{a_n^{(1)}}}{x_{-2}^{a_n^{(2)}} \cdots x_{-k+1}^{a_n^{(k-1)}} x_{-k}^{a_n^{(k)}}}\right\}. \tag{17}$$

From (17) and since $a_l^{(i)}$, $i \in \{1, \ldots, k\}$ are convergent sequences, the boundedness of x_n follows. □

2.3. *Case* $p = r + 1 = \frac{k}{k-1}$

Now we consider the case $p = r + 1 = \frac{k}{k-1}$.

Theorem 3. *Assume that* $p = r + 1 = \frac{k}{k-1}$. *Then Eq.* (2) *has positive unbounded solutions.*

Proof. Clearly formulae (12)–(16) also hold when $r = 1/(k-1)$ and from (13) the following identity follows

$$a_{n+1}^{(1)} - a_{n+1}^{(2)} - \cdots - a_{n+1}^{(k)} = a_n^{(1)} - a_n^{(2)} - \cdots - a_n^{(k)} = 1. \tag{18}$$

Now we prove that

$$a_n^{(1)} \to \infty \quad \text{as } n \to \infty. \tag{19}$$

From the proof of Theorem 2 we see that the sequence $a_n^{(1)}$ is increasing. On the other hand, the characteristic polynomial associated with Eq. (14) is

$$P(\lambda) = \frac{1}{k-1}(\lambda - 1)^2\big((k-1)\lambda^{k-2} + (k-2)\lambda^{k-3} + \cdots + 2\lambda + 1\big).$$

By Kakeya–Eneström theorem it follows that all the roots of the polynomial

$$(k-1)\lambda^{k-2} + (k-2)\lambda^{k-3} + \cdots + 2\lambda + 1 \tag{20}$$

lie in the unit disk.

Let $\hat{\lambda}_1, \ldots, \hat{\lambda}_t$ be different zeroes of polynomial (20). Then

$$a_n^{(1)} = c_1 + nc_2 + \sum_{j=1}^{t} Q_j(n)\hat{\lambda}_j^n,$$

for some $c_1, c_2 \in \mathbb{R}$ and some polynomials Q_j, $j = 1, \ldots, t$. Moreover since $a_n^{(1)}$ is increasing c_2 must be a positive number. Thus (19) is proved.

Now note that from (16) and (18) it follows that

$$x_n \geqslant \frac{x_{-1}^{a_n^{(1)}}}{x_{-2}^{a_n^{(2)}} \cdots x_{-k+1}^{a_n^{(k-1)}} x_{-k}^{a_n^{(k)}}} = x_{-1}\left(\frac{x_{-1}}{x_{-2}}\right)^{a_n^{(2)}} \cdots \left(\frac{x_{-1}}{x_{-k}}\right)^{a_n^{(k)}}. \tag{21}$$

If we choose the initial conditions such that $x_{-1} > \max\{x_{-2}, \ldots, x_{-k}\}$, $x_{-j} > 0$, $j = 2, \ldots, k$, then from (21) it follows that

$$x_n \geqslant x_{-1}\left(\frac{x_{-1}}{\max_{-k \leqslant i \leqslant -2} x_i}\right)^{\sum_{j=2}^{k} a_n^{(j)}} \to \infty,$$

as $n \to \infty$. Hence, all positive solutions of Eq. (2) with $x_{-1} > \max\{x_{-2}, \ldots, x_{-k}\}$ are unbounded, finishing the proof of the theorem. □

2.4. Case $r \leqslant (k-1)^{k-1}p^k/k^k$, $p < 1$

Here we study the case $r \leqslant (k-1)^{k-1}p^k/k^k$ and $p < 1$.

Theorem 4. *Assume that $r \leqslant (k-1)^{k-1}p^k/k^k$ and $p \in (0,1)$. Then every positive solution of Eq.* (2) *is bounded.*

Proof. Since $x_n \geqslant A$, $n \in \mathbb{N}_0$, we have that

$$x_n \leqslant \max\left\{A, \frac{x_{n-1}^p}{A^r}\right\}, \tag{22}$$

for $n = k, k+1, \dots$, where (x_n) is an arbitrary positive solution of (2).

Let y_n be the solution of the difference equation

$$y_n = \max\left\{A, \frac{y_{n-1}^p}{A^r}\right\}, \quad n = k, k+1, \dots, \tag{23}$$

such that $y_k = x_k$. Since the function

$$f(x) = \max\left\{A, \frac{x^p}{A^r}\right\}$$

is nondecreasing, by induction it follows that

$$x_n \leqslant y_n, \quad \text{for } n \geqslant k.$$

Now we prove that the solution of Eq. (23) is bounded.

Note that the function f is also concave on the interval $[A^{(r+1)/p}, \infty)$ (we use here the condition $p \in (0,1)$) it follows that there is a unique fixed point x^* of the equation $f(x) = x$, and the function f satisfies the condition

$$\big(f(x) - x\big)\big(x - x^*\big) < 0, \quad x \in (0,\infty) \setminus \{x^*\}. \tag{24}$$

Using (24) it is easy to see that if $y_k \in (0, x^*]$ the sequence $(y_n)_{n \geqslant k}$ is nondecreasing and bounded above by x^* and if $y_k \geqslant x^*$, it is nonincreasing and bounded below by x^*. Hence the sequence (y_n) is bounded and consequently the sequence (x_n). □

2.5. Case $r \leqslant (k-1)^{k-1}p^k/k^k$, $1 \leqslant p < r+1$, $r < 1/(k-1)$

This section is devoted to the case $r \leqslant (k-1)^{k-1}p^k/k^k$, $1 \leqslant p < r+1$, $r < 1/(k-1)$.

2.5.1. Case $r \leqslant (k-1)^{k-1}p^k/k^k$, $p = 1$, $A > 1$

In the case $r \leqslant (k-1)^{k-1}p^k/k^k$, $p = 1$, $A > 1$ we have the following result.

Theorem 5. *Assume that $r \leqslant (k-1)^{k-1}p^k/k^k$, $p = 1$ and $A > 1$. Then every positive solution of Eq.* (2) *is bounded.*

Proof. First note that if (x_n) is an arbitrary positive solution of (2) then inequality (22) holds, with $p = 1$.

Let y_n be the solution of the difference equation

$$y_n = A + \frac{y_{n-1}}{A^r}, \quad n = k, k+1, \dots, \tag{25}$$

such that $y_k = x_k$. As in the proof of Theorem 4 we have that $x_n \leqslant y_n$, for $n \geqslant k$.

Eq. (25) is a first order linear difference equation with constant coefficients. Hence by numerous known methods it follows that

$$\lim_{n\to\infty} y_n = \frac{A^{r+1}}{A^r - 1},$$

from which the boundedness of (y_n) follows and consequently the boundedness of (x_n), as claimed. □

2.5.2. Case $r \leqslant (k-1)^{k-1}p^k/k^k$, $p=1$, $A \in (0,1]$

Here we consider the case

$$r \leqslant (k-1)^{k-1}p^k/k^k, \quad p=1, \ A \in (0,1].$$

In dealing with Eq. (2) in this case we will use again "*Oachkatzlschwoif*" method.

Theorem 6. *Assume that $r \leqslant (k-1)^{k-1}p^k/k^k$, $p=1$ and $A \in (0,1]$. Then every positive solution of Eq.* (2) *is bounded.*

Proof. By applying consecutively Eq. (2), we obtain

$$\begin{aligned} x_n &= \max\left\{A, \frac{x_n}{x_{n-k}^r}\right\} = \max\left\{A, \frac{A}{x_{n-k}^r}, \frac{x_{n-2}}{x_{n-k}^r x_{n-k-1}^r}\right\} \\ &= \max\left\{A, \frac{A}{x_{n-k}^r}, \frac{A}{x_{n-k}^r x_{n-k-1}^r}, \frac{x_{n-3}}{x_{n-k}^r x_{n-k-1}^r x_{n-k-2}^r}\right\} \\ &= \cdots \\ &= \max\left\{A, \frac{A}{x_{n-k}^r}, \dots, \frac{A}{x_{n-k}^r \cdots x_{n-2k+2}^r}, \frac{x_{n-k}^{1-r}}{x_{n-k-1}^r \cdots x_{n-2k+1}^r}\right\}. \end{aligned}$$

Let

$$a_0^{(1)} = 1-r, \qquad a_0^{(i)} = r, \quad i = 2, \dots, k.$$

Further we have

$$\begin{aligned} x_n &= \max\left\{A, \frac{A}{x_{n-k}^r}, \dots, \frac{A}{x_{n-k}^r \cdots x_{n-2k+2}^r}, \frac{x_{n-k}^{a_0^{(1)}}}{x_{n-k-1}^{a_0^{(2)}} \cdots x_{n-2k+1}^{a_0^{(k)}}}\right\} \\ &= \max\left\{A, \frac{A}{x_{n-k}^r}, \dots, \frac{A}{x_{n-k}^r \cdots x_{n-2k+2}^r}, \frac{A^{a_0^{(1)}}}{x_{n-k-1}^{a_0^{(2)}} \cdots x_{n-2k+1}^{a_0^{(k)}}}, \frac{x_{n-k-1}^{a_0^{(1)}-a_0^{(2)}}}{x_{n-k-2}^{a_0^{(3)}} \cdots x_{n-2k+1}^{a_0^{(k)}} x_{n-2k}^{ra_0^{(1)}}}\right\} \\ &= \max\left\{A, \frac{A}{x_{n-k}^r}, \dots, \frac{A}{x_{n-k}^r \cdots x_{n-2k+2}^r}, \frac{A^{a_0^{(1)}}}{x_{n-k-1}^{a_0^{(2)}} \cdots x_{n-2k+1}^{a_0^{(k)}}}, \frac{x_{n-k-1}^{a_1^{(1)}}}{x_{n-k-2}^{a_1^{(2)}} \cdots x_{n-2k+1}^{a_1^{(k-1)}} x_{n-2k}^{a_1^{(k)}}}\right\} \\ &= \cdots \\ &= \max\left\{A, \frac{A}{x_{n-k}^r}, \dots, \frac{A}{x_{n-k}^r \cdots x_{n-2k+2}^r}, \frac{A^{a_0^{(1)}}}{x_{n-k-1}^{a_0^{(2)}} \cdots x_{n-2k+1}^{a_0^{(k)}}}, \dots, \right. \\ &\qquad \left. \frac{A^{a_{l-1}^{(1)}}}{x_{n-k-l}^{a_{l-1}^{(2)}} \cdots x_{n-2k+2-l}^{a_{l-1}^{(k)}}}, \frac{x_{n-k-l}^{a_{l-1}^{(1)}-a_{l-1}^{(2)}}}{x_{n-k-l-1}^{a_{l-1}^{(3)}} \cdots x_{n-2k+2-l}^{a_{l-1}^{(k)}} x_{n-2k+1-l}^{ra_{l-1}^{(1)}}}\right\}. \end{aligned} \tag{26}$$

By induction we obtain that sequences $a_l^{(i)}$, $i = 1, \dots, k$, satisfy the system

$$a_l^{(1)} = a_{l-1}^{(1)} - a_{l-1}^{(2)}, \qquad a_l^{(i)} = a_{l-1}^{(i+1)}, \quad i \in \{2, \dots, k-1\}, \qquad a_l^{(k)} = ra_{l-1}^{(1)}, \tag{27}$$

and consequently

$$a_l^{(1)} - a_{l-1}^{(1)} + ra_{l-k}^{(1)} = 0. \tag{28}$$

Moreover, as far as $a_l^{(1)} > 0$, it is decreasing, so that $a_l^{(i)}$, $i = 2, \dots, k$, are too. Hence there is nonnegative $\lim_{l\to\infty} a_l^{(1)} = a$, which must be equal to zero, and consequently $\lim_{l\to\infty} a_l^{(i)} = 0$, $i = 2, \dots, k$.

If there is a $k_0 \in \mathbb{N}$ such that $a_{k_0}^{(1)} \leqslant 0$ then it is clear that x_n is bounded (see the proof of Theorem 7).

Now assume that $a_l^{(1)}$ is a positive sequence.

Note that from (26) it follows that

$$x_n = \max\left\{A, \frac{A}{x_{n-k}^r}, \dots, \frac{A}{x_{n-k}^r \cdots x_{n-2k+2}^r}, \frac{A^{a_0^{(1)}}}{x_{n-k-1}^{a_0^{(2)}} \cdots x_{n-2k+1}^{a_0^{(k)}}}, \dots, \frac{A^{a_{n-k}^{(1)}}}{x_{-1}^{a_{n-k}^{(2)}} \cdots x_{-k+1}^{a_{n-k}^{(k)}}}, \frac{x_{-1}^{a_{n-k+1}^{(1)}}}{x_{-2}^{a_{n-k+1}^{(2)}} \cdots x_{-k}^{a_{n-k+1}^{(k)}}}\right\}. \tag{29}$$

Since the sequences $a_l^{(i)}$, $i = 1, 2, \dots, k$, converge to zero in this case, we have

$$\frac{A^{a_l^{(1)}}}{x_{n-l-k-1}^{a_l^{(2)}} \cdots x_{n-l-2k+1}^{a_l^{(k)}}} \leqslant \frac{1}{A^{\sum_{i=2}^k a_l^{(i)} - a_l^{(1)}}} \asymp 1, \tag{30}$$

for $n - 2k + 1 \geqslant l$.

Finally note that

$$\lim_{n\to\infty} \frac{A^{a_{n-k+1-i}^{(1)}}}{x_{i-2}^{a_{n-k+1-i}^{(2)}} \cdots x_{i-k}^{a_{n-k+1-i}^{(k)}}} = \lim_{n\to\infty} \frac{x_{-1}^{a_{n-k+1}^{(1)}}}{x_{-2}^{a_{n-k+1}^{(2)}} \cdots x_{-k}^{a_{n-k+1}^{(k)}}} = 1, \tag{31}$$

for $i = 1, \dots, k-1$.

From (29), (30) and (31) the boundedness of x_n follows, as claimed. □

2.6. Case $p^k \in (0, rk^k/(k-1)^{k-1})$

Here we investigate the boundedness of the positive solutions to Eq. (2) for the case $p^k \in (0, rk^k/(k-1)^{k-1})$. The following result completely describes the boundedness of positive solutions to Eq. (2) in this case.

Theorem 7. *Assume that $p^k \in (0, rk^k/(k-1)^{k-1})$. Then all positive solutions to Eq.* (2) *are bounded.*

Proof. By repeating use of the recurrence relation in Eq. (2), we obtain the following chain of equalities

$$\begin{aligned}
x_n &= \max\left\{A, \frac{x_{n-1}^p}{x_{n-k}^r}\right\} \\
&= \max\left\{A, \max\left\{\frac{A}{x_{n-k}^{r/p}}, \frac{x_{n-2}^p}{x_{n-k}^{r/p} x_{n-k-1}^r}\right\}^p\right\} = \max\left\{A, \frac{A^p}{x_{n-k}^r}, \left\{\frac{x_{n-2}}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}}\right\}^{p^2}\right\} \\
&= \max\left\{A, \left\{\frac{A}{x_{n-k}^{r/p}}\right\}^p, \left\{\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}}\right\}^{p^2}, \left\{\frac{x_{n-3}^p}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p} x_{n-k-2}^r}\right\}^{p^2}\right\} \\
&= \max\left\{A, \left\{\frac{A}{x_{n-k}^{r/p}}\right\}^p, \left\{\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}}\right\}^{p^2}, \left\{\frac{x_{n-3}}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}}\right\}^{p^3}\right\} \\
&= \max\left\{A, \left\{\frac{A}{x_{n-k}^{r/p}}\right\}^p, \left\{\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}}\right\}^{p^2}, \left\{\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}}\right\}^{p^3}, \left\{\frac{x_{n-4}}{x_{n-k}^{r/p^4} x_{n-k-1}^{r/p^3} x_{n-k-2}^{r/p^2} x_{n-k-3}^{r/p}}\right\}^{p^4}\right\} \\
&= \cdots \\
&= \max\left\{A, \left\{\frac{A}{x_{n-k}^{r/p}}\right\}^p, \left\{\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}}\right\}^{p^2}, \left\{\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}}\right\}^{p^3}, \dots, \left\{\frac{A}{x_{n-k}^{r/p^{k-1}} x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p}}\right\}^{p^{k-1}},\right. \\
&\qquad \left.\left\{\frac{x_{n-k}^p}{x_{n-k}^{r/p^{k-1}} x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p} x_{n-(2k-1)}^r}\right\}^{p^{k-1}}\right\} \\
&= \max\left\{A, \left\{\frac{A}{x_{n-k}^{r/p}}\right\}^p, \left\{\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}}\right\}^{p^2}, \left\{\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}}\right\}^{p^3}, \dots, \left\{\frac{A}{x_{n-k}^{r/p^{k-1}} x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p}}\right\}^{p^{k-1}},\right. \\
&\qquad \left.\left\{\frac{x_{n-k}^{p-\frac{r}{p^{k-1}}}}{x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p} x_{n-(2k-1)}^r}\right\}^{p^{k-1}}\right\}.
\end{aligned} \tag{32}$$

First assume that $p^k \leqslant r$. From (32) it follows that

$$x_n \leqslant \max\left\{A, \frac{1}{A^{r-p}}, \frac{1}{A^{r+rp-p^2}}, \ldots, \frac{1}{A^{r\sum_{j=0}^{k-2} p^j - p^{k-1}}}, \frac{1}{A^{r\sum_{j=0}^{k-1} p^j - p^k}}\right\} < \infty$$

for $n \geqslant 2k-1$, where we have used the fact that $x_n \geqslant A$ for $n \in \mathbb{N}_0$. Therefore (x_n) is bounded in this case.

Now we assume $p^k > r$. This implies that

$$p - \frac{r}{p^{k-1}} =: p - a_0^{(0)} > 0.$$

We continue the development of x_n obtained in (32). We have

$$\begin{aligned}
x_n &= \max\left\{A, \left\{\frac{A}{x_{n-k}^{r/p}}\right\}^p, \left\{\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}}\right\}^{p^2}, \left\{\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}}\right\}^{p^3}, \ldots, \left\{\frac{A}{\prod_{j=0}^{k-2} x_{n-k-j}^{a_0^{(j)}}}\right\}^{p^{k-1}},\right. \\
&\qquad \left.\left\{\frac{x_{n-k}^{p-a_0^{(0)}}}{(\prod_{j=1}^{k-2} x_{n-k-j}^{a_0^{(j)}}) x_{n-(2k-1)}^r}\right\}^{p^{k-1}}\right\} \\
&= \max\left\{A, \left\{\frac{A}{x_{n-k}^{r/p}}\right\}^p, \left\{\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}}\right\}^{p^2}, \left\{\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}}\right\}^{p^3}, \ldots, \left\{\frac{A}{\prod_{j=0}^{k-2} x_{n-k-j}^{a_0^{(j)}}}\right\}^{p^{k-1}},\right. \\
&\qquad \left.\left\{\frac{x_{n-k}}{(\prod_{j=1}^{k-2} x_{n-k-j}^{a_0^{(j)}/(p-a_0^{(0)})}) x_{n-(2k-1)}^{r/(p-a_0^{(0)})}}\right\}^{p^{k-1}(p-a_0^{(0)})}\right\} \\
&= \max\left\{A, \left\{\frac{A}{x_{n-k}^{r/p}}\right\}^p, \left\{\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}}\right\}^{p^2}, \ldots, \left\{\frac{A}{\prod_{j=0}^{k-2} x_{n-k-j}^{a_0^{(j)}}}\right\}^{p^{k-1}}, \left\{\frac{A}{\prod_{j=0}^{k-2} x_{n-k-1-j}^{a_1^{(j)}}}\right\}^{p^{k-1}(p-a_0^{(0)})},\right. \\
&\qquad \left.\left\{\frac{x_{n-k-1}^{p-a_1^{(0)}}}{\prod_{j=1}^{k-2} x_{n-k-1-j}^{a_1^{(j)}} x_{n-2k}^r}\right\}^{p^{k-1}(p-a_0^{(0)})}\right\} \\
&= \cdots \\
&= \max\left\{A, \left\{\frac{A}{x_{n-k}^{r/p}}\right\}^p, \left\{\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}}\right\}^{p^2}, \left\{\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}}\right\}^{p^3}, \ldots, \left\{\frac{A}{\prod_{j=0}^{k-2} x_{n-k-m-j}^{a_m^{(j)}}}\right\}^{p^{k-1}\prod_{i=0}^{m-1}(p-a_i^{(0)})},\right. \\
&\qquad \left.\left\{\frac{x_{n-k-m}^{p-a_m^{(0)}}}{(\prod_{j=1}^{k-2} x_{n-k-m-j}^{a_m^{(j)}}) x_{n-2k+1-m}^r}\right\}^{p^{k-1}\prod_{i=0}^{m-1}(p-a_i^{(0)})}\right\}
\end{aligned} \tag{33}$$

for each $k \in \mathbb{N} \setminus \{1\}$ and every $n \geqslant 2k+m-1$, where the sequences $a_m^{(j)}$, $j \in \{0, 1, \ldots, k-2\}$, are defined by

$$a_{m+1}^{(0)} = \frac{a_m^{(1)}}{p - a_m^{(0)}}, \quad a_{m+1}^{(1)} = \frac{a_m^{(2)}}{p - a_m^{(0)}}, \quad \ldots, \quad a_{m+1}^{(k-3)} = \frac{a_m^{(k-2)}}{p - a_m^{(0)}}, \quad a_{m+1}^{(k-2)} = \frac{r}{p - a_m^{(0)}}, \tag{34}$$

with

$$a_0^{(j)} = \frac{r}{p^{k-1-j}}, \quad j \in \{0, 1, \ldots, k-2\}.$$

Recall that $p^k > r$, so that $a_0^{(0)} < p$. Now suppose $a_m^{(0)} < p$ for every $m \in \mathbb{N}_0$. Since

$$a_0^{(j)} = \frac{r}{p^{k-1-j}} < \frac{\frac{r}{p^{k-1-(j+1)}}}{p - \frac{r}{p^{k-1}}} = a_1^{(j)}, \quad j = 0, 1, \ldots, k-2,$$

by using (34), it is easy to see that the sequences $a_m^{(j)}$, $j \in \{0, 1, \ldots, k-2\}$ are strictly increasing. From (34), we also have that

$$a_{m+1}^{(0)} = \frac{r}{(p - a_m^{(0)})(p - a_{m-1}^{(0)}) \cdots (p - a_{m-k+2}^{(0)})}, \quad m \geqslant k-2.$$

Hence, if $a_m^{(0)} < p$ for every $m \in \mathbb{N}_0$ we have that there is finite limit

$$\lim_{m\to\infty} a_m^{(0)} = x^* \in (0, p],$$

and that x^* is a solution to the equation

$$f(x) = x(p-x)^{k-1} - r = 0.$$

Since

$$f'(x) = (p-x)^{k-2}(p-kx)$$

we see that the function attains its maximum on the interval $[0, p]$, at the point $x = p/k$. On the other hand, we have that

$$f(p/k) = \frac{(k-1)^{k-1}}{k^k} p^k - r = \frac{(k-1)^{k-1}}{k^k}\left(p^k - r\frac{k^k}{(k-1)^{k-1}}\right). \tag{35}$$

From (35), since $p^k < r\frac{k^k}{(k-1)^{k-1}}$, $f(0) = f(p) = -r < 0$, it follows that the equation $f(x) = 0$ has no solutions on the interval $(0, p]$, which is a contradiction.

Hence, there is a $k_0 \in \mathbb{N}$ such that

$$a_{k_0-1}^{(0)} < p \quad \text{and} \quad a_{k_0}^{(0)} \geqslant p.$$

From this and (33) with $m = k_0$, it follows that

$$\begin{aligned} x_n = \max\Bigg\{ & A, \left\{\frac{A}{x_{n-k}^{r/p}}\right\}^p, \left\{\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}}\right\}^{p^2}, \left\{\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}}\right\}^{p^3}, \ldots, \left\{\frac{A}{\prod_{j=0}^{k-2} x_{n-k-k_0-j}^{a_{k_0}^{(j)}}}\right\}^{p^{k-1}\prod_{i=0}^{k_0-1}(p-a_i^{(0)})}, \\ & \left\{\frac{x_{n-k-k_0}^{p-a_{k_0}^{(0)}}}{(\prod_{j=1}^{k-2} x_{n-k-k_0-j}^{a_{k_0}^{(j)}}) x_{n-2k+1-k_0}^{r}}\right\}^{p^{k-1}\prod_{i=0}^{k_0-1}(p-a_i^{(0)})} \Bigg\} \\ \leqslant \max\Bigg\{ & A, \frac{1}{A^{r-p}}, \frac{1}{A^{r+rp-p^2}}, \ldots, \frac{1}{A^{p^{k-1}\prod_{i=0}^{k_0-1}(p-a_i^{(0)})(\sum_{j=0}^{k-2} a_{k_0}^{(j)} - 1)}}, \frac{1}{A^{p^{k-1}\prod_{i=0}^{k_0-1}(p-a_i^{(0)})(\sum_{j=0}^{k-2} a_{k_0}^{(j)} + r - p)}} \Bigg\} < \infty \end{aligned}$$

for $n \geqslant 2k + k_0 - 1$ (here we again use the fact that $x_n \geqslant A$ for $n \in \mathbb{N}_0$).

The last expression is an upper bound for the sequence x_n, finishing the proof of the theorem. □

At the moment we are not able to solve the boundedness in the case

$$r \leqslant (k-1)^{k-1} p^k / k^k, \qquad 1 < p < r+1, \qquad r < 1/(k-1). \tag{36}$$

However, we pose the following conjecture

Conjecture 1. *Assume that parameters A, p and r satisfy conditions in* (36). *Then every positive solution of Eq.* (2) *is bounded.*

Table 1 summarizes the results in this paper.

Table 1

Case	Boundedness character of positive solutions
$p^k \geqslant r\frac{k^k}{(k-1)^{k-1}}$, $p > \frac{k}{k-1}$	there are unbounded solutions (Theorem 1)
$p^k > r\frac{k^k}{(k-1)^{k-1}}$, $p = \frac{k}{k-1}$	there are unbounded solutions (Theorem 1)
$r + 1 < p < \frac{k}{k-1}$	there are unbounded solutions (Theorem 1)
$p = r + 1$ and $r \in (0, 1/(k-1))$	all solutions are bounded (Theorem 2)
$p = r + 1 = k/(k-1)$	there are unbounded solutions (Theorem 3)
$p^k \geqslant r\frac{k^k}{(k-1)^{k-1}}$, $p \in (0, 1]$	all solutions are bounded (Theorems 4–6)
$p^k \in (0, rk^k/(k-1)^{k-1})$	all solutions are bounded (Theorem 7)
$p^k \geqslant r\frac{k^k}{(k-1)^{k-1}}$, $1 < p < r + 1$, $r < 1/(k-1)$	not solved (Conjecture 1)

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